

Local Cloning of Entangled Qubits

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We discuss the exact cloning of orthogonal but entangled qubits under local operations and classical communication. The amount of entanglement necessary in blank copy is obtained for various cases. Surprisingly this amount is more than 1 ebit for certain set of two nonmaximal but equally entangled states of two qubits system. To clone any three two qubits Bell states at least $\log_2 3$ ebit is necessary.

PACS numbers: 03.67.Hk, 03.67.Mn

I. INTRODUCTION

Classical states can always be cloned perfectly. But the quantum no cloning theorem [1] prohibits exact cloning of nonorthogonal states. However, orthogonal quantum states can always be cloned if one can perform a operation on the entire system.

A common scenario in quantum information processing is where a multipartite entangled state is distributed among a number of spatially separated parties. Each of these parties are able to perform only local operations on the subsystem they possess and can send only classical information to each other. This is known as LOCC (Local operation and classical communication). If we restrict ourselves only to LOCC, further restrictions on cloning apply. For example, the very obvious first restriction will be; an entangled blank state is needed to clone an entangled state. Moreover, entanglement of blank state should at least be equal to the entanglement of the state to be cloned, or else entanglement of the entire system will increase under LOCC which is impossible. However, with a sufficient supply of entanglement; entangled states can be cloned by LOCC. For example, any arbitrary set of orthogonal states of two qubits can be cloned with the help of 3 ebit. Any set of two orthogonal states need only 2 ebit.

The concept of entanglement cloning under LOCC was first considered by Ghosh *et. al.* [2] where it was shown that for LOCC cloning of two orthogonal Bell states and four orthogonal Bell states, 1-ebit and 2-ebit of entanglement is necessary and sufficient. Later many works have been done in this direction [3, 4], which involve maximally entangled states. In this paper, we consider cloning of arbitrary but equally entangled orthogonal states under LOCC and the following interesting results are found:

- (i) $\log_2 3$ ebit in the blank copy is necessary to clone any three Bell states.
- (ii) Local exact cloning of any two orthogonal entangled states is not possible with the help of same entanglement unless the states are maximally entangled.
- (iii) Even a maximally entangled state of two qubits may not help as blank copy for cloning a set of two orthogonal nonmaximal equally entangled states if these states lie in the same plane.

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II. CLONING BELL STATES

The four Bell states are given as:

$$|B_{mn}\rangle = \frac{1}{\sqrt{2}} \sum_{j=0}^1 e^{2\pi i j n/2} |j\rangle |j \oplus m\rangle, n, m = 0, 1. \quad (1)$$

where one qubit is held by Alice and the other is held by Bob.

In a very elegant way, Ghosh *et. al.* [2] has shown that any two Bell states can be cloned with the help of 1 ebit, whereas to copy all the 4 Bell states, one needs at least 2 ebit of entanglement in the blank copy. Recently, Owari and Hayashi [4] have shown that any three Bell states cannot be cloned if one ebit free entanglement is supplied as resource. We, in this section, by entanglement considerations, not only prove the same but also provide the necessary entanglement resource for such a cloning.

To obtain the necessary amount of entanglement needed in the blank copy for local cloning (now and onwards by **‘local cloning’** or **‘cloning’** we will mean **‘exact cloning under LOCC’**) of three Bell states, we will make use of the fact that the relative entropy of entanglement cannot be increased by any LOCC operation. The relative entropy of entanglement for a bipartite quantum state ρ is defined by [5]:

$$E_R(\rho) = \min_{\sigma \in D(H)} S(\rho \| \sigma)$$

Here D is the set of all separable states on the Hilbert space H on which ρ is defined and $S(\rho \| \sigma)$ (the relative entropy of ρ to σ) is given by $S(\rho \| \sigma) \equiv \text{tr}(\rho \log_2 \rho) - \text{tr}(\rho \log_2 \sigma)$.

Let $\rho_1 \in H^1$ and $\rho_2 \in H^2$ be two quantum states and let $E_R(\rho_1) = S(\rho_1 \| \sigma_1)$, $E_R(\rho_2) = S(\rho_2 \| \sigma_2)$; *i.e.* $\sigma_1 \in H_1$ and $\sigma_2 \in H_2$ are the two separable states which minimize the relative entropies of ρ_1 and ρ_2 respectively. Let σ be the separable state belonging to the Hilbert space $H_1 \otimes H_2$ which minimizes the relative entropy of $\rho_1 \otimes \rho_2$. Then:

$$E_R(\rho_1 \otimes \rho_2) \leq S(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) \quad (2)$$

equality holds when $\sigma_1 \otimes \sigma_2 = \sigma$.

It was known [6]

$$S(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = S(\rho_1 \| \sigma_1) + S(\rho_2 \| \sigma_2) \quad (3)$$

hence

$$E_R(\rho_1 \otimes \rho_2) \leq S(\rho_1 \| \sigma_1) + S(\rho_2 \| \sigma_2) \quad (4)$$

i.e.

$$E_R(\rho_1 \otimes \rho_2) \leq E_R(\rho_1) + E_R(\rho_2) \quad (5)$$

If cloning of three Bell states (*e.g.* $|B_{00}\rangle, |B_{01}\rangle, |B_{10}\rangle$) is possible with a known entangled state (say $|B\rangle$) as blank copy (resource), then the following state

$\frac{1}{3} [|B_{00}^{\otimes 2}\rangle \langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle \langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle \langle B_{10}^{\otimes 2}|]$ along with the blank state $|B\rangle$ given as the input to the cloner will provide the output as:

$$\begin{aligned} \rho_{in} & \left(= \frac{1}{3} [|B_{00}^{\otimes 2}\rangle \langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle \langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle \langle B_{10}^{\otimes 2}|] \otimes |B\rangle \langle B| \right) \\ & \longrightarrow \rho_{out} \left(= \frac{1}{3} [|B_{00}^{\otimes 3}\rangle \langle B_{00}^{\otimes 3}| + |B_{01}^{\otimes 3}\rangle \langle B_{01}^{\otimes 3}| + |B_{10}^{\otimes 3}\rangle \langle B_{10}^{\otimes 3}|] \right) \end{aligned}$$

We now compare the relative entropies of entanglement of ρ_{in} and ρ_{out} .

From inequality (5), we have

$$E_R(\rho_{in}) \leq E_R \left(\frac{1}{3} [|B_{00}^{\otimes 2}\rangle \langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle \langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle \langle B_{10}^{\otimes 2}|] \right) + E_R(|B\rangle \langle B|)$$

As $E_R\left(\frac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle\langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle\langle B_{10}^{\otimes 2}|]\right) \leq 2 - \log_2 3$ [7], hence:

$$E_R(\rho_{in}) \leq 2 - \log_2 3 + E_R(|B\rangle\langle B|)$$

At least 2 ebit of entanglement can be distilled from ρ_{out} [8] and the distillable entanglement is bounded above by E_R , hence

$$E_R(\rho_{out}) \geq 2.$$

But relative entropy of entanglement cannot increase under LOCC, and in the output we have at least 2 ebit of relative entropy of entanglement, hence, in order to make cloning possible, $\log_2 3$ ebit is necessary in the blank state. Any two qubit state (even a two qubit maximally entangled state) cannot provide this necessary amount of entanglement.

III. CLONING ARBITRARY ENTANGLED STATES

Any two equally entangled orthogonal states can lie either in same plane:

(I)

$$|\Psi_1\rangle = a|00\rangle + b|11\rangle$$

$$|\Psi_2\rangle = b|00\rangle - a|11\rangle$$

or in different planes:

(II)

$$|\Psi_1\rangle = a|00\rangle + b|11\rangle$$

$$|\Psi_3\rangle = a|01\rangle + b|10\rangle$$

where a,b are real and unequal and $a^2 + b^2 = 1$.

In both the cases, if one provide two entangled states, each having same entanglement as in the original one, cloning will be trivially possible. Here we investigate the nontrivial case when a single entangled qubit state is supplied as blank copy.

Case(I)

Suppose there exists a cloning machine which can clone $|\Psi_1\rangle$ and $|\Psi_2\rangle$ when a pure entangled qubit state $|\Phi\rangle (= c|00\rangle + d|11\rangle; c^2 + d^2 = 1)$ is supplied to it as blank copy. Let us supply an equal mixture of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ together with the blank state $|\Phi\rangle$ to it; *i.e.* the state input to the cloner is:

$$\rho_{in} = \left[\frac{1}{2}P(|\Psi_1\rangle) + \frac{1}{2}P(|\Psi_2\rangle) \right] \otimes P(|\Phi\rangle) \quad (6)$$

The output of the cloner:

$$\rho_{out} = \frac{1}{2}P[|\Psi_1\rangle \otimes |\Psi_1\rangle] + \frac{1}{2}P[|\Psi_2\rangle \otimes |\Psi_2\rangle] \quad (7)$$

For proving impossibility of such a cloner, we make use of the fact that Negativity, of a bipartite quantum state ρ , $N(\rho)$ cannot increase under LOCC [10]. $N(\rho)$ is given by [11]

$$N(\rho) \equiv \|\rho^{T_B}\| - 1 \quad (8)$$

where ρ^{T_B} is the partial transpose with respect to system B and $\|\dots\|$ denotes the trace norm which is defined as,

$$\|\rho^{T_B}\| = \text{tr}(\sqrt{\rho^{T_B^\dagger} \rho^{T_B}}) \quad (9)$$

The negativity of the input state ρ_{in} is

$$N(\rho_{in}) = 2cd \leq 1$$

whereas, the negativity of the output is

$$N(\rho_{out}) = 4a^2b^2 + 4\sqrt{a^2b^2(a^2 - b^2)^2}$$

The above cloning will not be possible as long as,

$$cd < 2a^2b^2 + 2\sqrt{a^2b^2(a^2 - b^2)^2} \quad (10)$$

The above inequality has some interesting features, but the most significant feature is:

‘Even a maximally entangled state of two qubits cannot help as blank copy for a large number of pairs of nonmaximally entangled state belonging to this class’(see the graph below). Numerical calculations show that this is the case for $0.230 \leq a \leq 0.973$ (except for $a = \frac{1}{\sqrt{2}}$). This is surprising as recently Kay and Ericsson [12] have given a protocol by which all the pairs of states lying in different planes (II) can be cloned with the help of 1 free ebit.

Other important features are: (a) For $a = b = c = d = \frac{1}{\sqrt{2}}$ the above inequality becomes an equality. This is consistent with an earlier finding [2] that two maximally entangled bipartite state can be cloned with 1 free ebit.

(b) Inequality (10) holds even for $c = a \neq d = b$ (see the graph below). This in turn implies that same amount of entanglement (as in the state to be cloned) cannot help as blank copy, for any pair of nonmaximally entangled states.

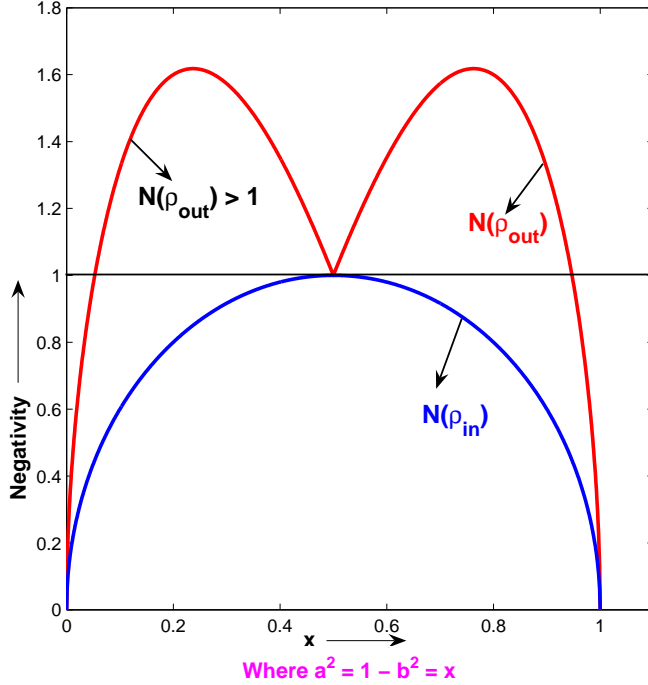


Figure 1: The Negativity of the output is more than that of the input except for maximally entangled ones.

Case (II)

This time we suppose that our cloning machine can clone $|\Psi_1\rangle$ and $|\Psi_3\rangle$ if a pure entangled state $|\Phi\rangle (= c|00\rangle + d|11\rangle; c^2 + d^2 = 1)$ is used as blank copy.

Let the state supplied to this machine be:

$$\rho_{in} = \frac{1}{2} [P(|\Psi_1\rangle) + P(|\Psi_3\rangle)] \otimes P[|\Phi\rangle]$$

We then have output of the cloner as:

$$\rho_{out} = \frac{1}{2}P[|\Psi_1\rangle \otimes |\Psi_1\rangle] + \frac{1}{2}P[|\Psi_3\rangle \otimes |\Psi_3\rangle]$$

Putting for $|\Psi_1\rangle$ $|\Psi_3\rangle$ and $|\phi\rangle$ in the expression for ρ_{in} and ρ_{out} and making use of equations (8) and (9), we get:

$$N(\rho_{in}) = 2cd \leq 1$$

$$N(\rho_{out}) = 2\sqrt{2(a^6b^2 + a^2b^6)}$$

From nonincrease of negativity under LOCC it follows that as long as

$$cd < \sqrt{2(a^6b^2 + a^2b^6)} \quad (11)$$

the above cloning is not possible.

(a) $a = b = c = d = \frac{1}{\sqrt{2}}$ turns this inequality into an equality. This again is consistent with [2].

(b) If we put $c = a \neq d = b$ in the above inequality, *i.e* if we use same amount of entanglement (as in original states) then too cloning remains impossible as can be seen from the following graph:

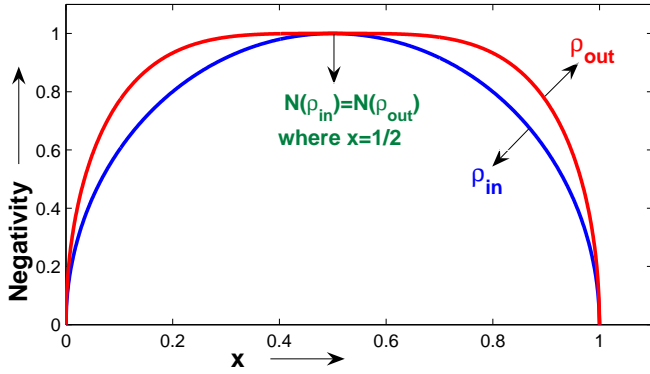


Figure 2 : Negativity of output is more than that of the input except for maximally entangled ones.

(c) Here too the inequality (10) shows that for any entanglement in the original states, except the maximally ones, the necessary entanglement in the blank copy is always higher. As an example, for $a = \sqrt{0.3}$, (*i.e.* entanglement of the state to be cloned 0.8813), as long as $c < \sqrt{0.42}$, (*i.e.* entanglement of blank copy < 0.9815), cloning is not possible.

IV. CONCLUSION

In this paper we addressed the problem of LOCC cloning for entangled states. To clone three Bell states, one need at least $\log_2 3$ ebit in the blank state. So any two qubit state (pure or mixed) cannot serve this purpose. We have also shown the blank state needed should have more free entanglement than the original ones, for cloning any pair of nonmaximal but equally entangled orthogonal states. The necessary amount of entanglement in the blank state for such cloning to be possible is given by inequalities (10) and (11). Interestingly this necessary amount is more than 1 ebit for certain set of nonmaximal but equally entangled states contrary to certain other sets for which 1 ebit can serve as blank copy.

Acknowledgments

The authors acknowledge G.Kar for valuable suggestions. R.R acknowledges the support by CSIR, Government of India, New Delhi.

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 - [8] The control-not operation C is defined as $C|i\rangle\otimes|j\rangle = |i\rangle\otimes|j\oplus i\rangle$, and the bilateral control-not operation (BXOR) defined on bipartite system as, \mathcal{B} is $\mathcal{B}|i\rangle_{A1}|r\rangle_{B1}\otimes|j\rangle_{A2}|s\rangle_{B2} = |i\rangle_{A1}|r\rangle_{B1}\otimes|j\oplus i\rangle_{A2}|s\oplus r\rangle_{B2}$. Denote $\mathcal{B}(m, n)$ as the BXOR operation performed on the m th pair (source) and the n th pair (target), the following operation will give, $\mathcal{B}(1, 3)\mathcal{B}(2, 3)|B_{mn}^{\otimes 3}\rangle = |B_{m,0}^{\otimes(2)}\rangle|B_{\oplus 3m,n}\rangle$ [9]. If this operation is applied on ρ_{out} , one get $\frac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}| \otimes [|B_{00}\rangle\langle B_{00}| + |B_{01}\rangle\langle B_{01}|] + |B_{10}^{\otimes 2}\rangle\langle B_{10}^{\otimes 2}| \otimes |B_{10}\rangle\langle B_{10}|]$. If Alice and Bob do the measurement in $|0\rangle, |1\rangle$ basis on the third copy and communicate, the results will be either correlated or anticorrelated. When they they are correlated the first two copies are in $|B_{00}\rangle$, and in other case they are in state $|B_{10}\rangle$, therefore distilling two ebits in this process.
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